Integration Worked Examples

Questions taken from Ex 20C of *Understanding Pure Mathematics*
by Sadler and Thorning, Oxford University Press, 2002

1. $\int (3x^2 - 6)dx$

This is solved using term-by term integration. Add 1 to the power and divide by the new power. Always add the constant.

$$\int (3x^2 - 6)dx \;=\; x^3 - 6x + c$$

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2. $\int \tan x \, dx$

Although this is a standard integral that can be found in the formula book, working it through from first principles demonstrates the use of the substitution method. We need to use a basic trig identity first, as is sometimes the case.

$$\int \tan x \, dx \;=\; \int \frac{\sin x}{\cos x} dx \qquad \text{Let } u = \cos x\,, \qquad \frac{du}{dx} = -\sin x \qquad \therefore\; dx = -\frac{du}{\sin x}$$

$$\therefore \int \tan x \, dx \;=\; -\int \frac{\sin x}{u}\frac{du}{\sin x} \;=\; -\int \frac{du}{u} \;=\; -\ln|u| + c \;= -\ln|\cos x| \; + c$$

Or alternatively expressed: $\int \tan x \, dx \;=\; \ln\left|\frac{1}{\cos x}\right| + c \;=\; \ln|sec\ x| + c$

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3. $\int \sin 2x \, dx$

The rule of thumb is that you integrate the basic function and divide through by the derivative of that function's argument (i.e. what's in the brackets.) If unsure, use the method in Q.2.

Let $u = 2x,\qquad \frac{du}{dx} = 2 \qquad \therefore\; dx = \frac{du}{2}$

$$\therefore \int \sin 2x \, dx \;=\; \int \sin u \,\frac{du}{2} \;=\; \frac{-\cos u}{2} + c \;=\; \frac{-\cos 2x}{2} + c$$

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4. $\int \frac{2}{x^2 - 1} dx$

The key to this question is to spot the difference between two squares in the denominator. This means we can split this fraction into two simpler fractions using the method of partial fractions.

$$\int \frac{2}{x^2-1}dx = \int \frac{A}{(x-1)} + \frac{B}{(x+1)}dx = \int \frac{1}{(x-1)} - \frac{1}{(x+1)}dx$$ Check you can do this step

$$\therefore \int \frac{2}{x^2-1}dx = \ln|x-1| - \ln|x+1| + c = \ln\left|\frac{(x-1)}{(x+1)}\right| + c$$

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5. $\int 3e^x dx$

Remember that the integral and derivative of e^x is always e^x

$$\int 3e^x dx = 3e^x + c$$

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6. $\int e^{3x} dx$

Use the rule in Q.3 again

$$\int e^{3x} dx = \frac{e^{3x}}{3} + c$$

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7. $\int \frac{1}{x^2} dx$

This is solved by rewriting the expression in index form which can then be integrated directly. Add 1 to the power and divide by the new power.

$$\int \frac{1}{x^2}dx = \int x^{-2}dx = -x^{-1} + c = -\frac{1}{x} + c$$

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8. $\int \frac{1}{x} dx = \ln|x| + c$ A standard integral which needs memorising

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9. $\int \cos^3 2x \, dx$

To integrate an ODD powered trig function, the best strategy is to use trig identities in order to generate two terms each of which can be integrated. In the last line, use the recognition method to spot that *cos2x* is the derivative of *sin2x* so the integral of this must be based on $sin^3 2x$

$$\therefore \int \cos^3 2x \, dx = \int \cos 2x \cos^2 2x \, dx = \int \cos 2x \, (1 - \sin^2 2x)$$

$$\therefore \int \cos^3 2x \, dx = \int \cos 2x - \cos 2x \sin^2 2x \; dx = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + c$$

10. $\int \frac{2}{(2x-3)^3}\,dx$

Recognise that 2 in the numerator is the derivative of 2x in the denominator, in which case we could guess the answer. Without guessing, we use substitution again.

$$\int \frac{2}{(2x-3)^3}\,dx \;=\; \int 2(2x-3)^{-3}dx \qquad \text{Let } \; u = 2x-3, \qquad \frac{du}{dx} = 2 \quad \therefore\; dx = \frac{du}{2}$$

$$\therefore \int \frac{2}{(2x-3)^3}\,dx \;=\; \int 2u^{-3}\frac{du}{2} \;=\; \int u^{-3}du$$

$$\therefore \int \frac{2}{(2x-3)^3}\,dx \;=\; -\frac{1}{2u^2} + c \;=\; -\frac{1}{2(2x-3)^2} + c$$

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11. $\int \frac{2}{(2x+3)}\,dx$

The denominator is a linear power of x so this must be a natural logs solution. Recognise the form of the solution but be careful with the final coefficient. Use substitution to be sure.

$$\text{Let } \; u = 2x+3, \qquad \frac{du}{dx} = 2 \qquad \therefore\; dx = \frac{du}{2}$$

$$\therefore \int \frac{2}{(2x+3)}\,dx = \int \frac{2}{u}\frac{du}{2} = \int \frac{du}{u} \;=\; \ln|u| + c \;=\; \ln|2x+3| + c$$

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12. $\int \frac{1}{\sqrt{(1-x^2)}}\,dx$

Although this is a standard integral, it's very useful to see how to derive it as this could be asked of you in an exam. The trick here is to use a trig substitution (which will almost certainly be given to you). We'll use the well known trig identity again to simplify the algebra.

$$\text{Let } \; x = \sin u \;\; \therefore\; u = \sin^{-1} x\,, \qquad \frac{dx}{du} = \cos u \qquad \therefore\; dx = \cos u\, du$$

$$\therefore \int \frac{1}{\sqrt{(1-x^2)}}\,dx \;=\; \int \frac{1}{\sqrt{(1-\sin^2 u)}}\cos u\, du \;=\; \int \frac{1}{\sqrt{(\cos^2 u)}}\cos u\, du \;=\; \int du$$

$$\therefore \int \frac{1}{\sqrt{(1-x^2)}}\,dx \;= u + c \;=\; \sin^{-1} x + c$$

13. $\int x(x-1)^6 dx$

We could expand this expression using the binomial theorem but it will get very messy. Suppose the power is 56 rather than 6 ! Let's try a substitution. Integration by parts would also work but is slightly more complicated.

Let $u = x - 1 \quad \therefore \; x = u + 1, \qquad \frac{du}{dx} = 1 \quad \therefore \; dx = du$

$$\therefore \int x(x-1)^6 dx = \int (u+1)u^6 du = \int u^7 + u^6 du$$

$$\therefore \int x(x-1)^6 dx = \frac{u^8}{8} + \frac{u^7}{7} + c = \frac{u^7}{56}(7u+8) + c = \frac{1}{56}(x-1)^7(7x+1) + c$$

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14. $\int \frac{1}{4+x^2} dx$

This is a standard integral which can be found in the formula book with the answer being in the form $\tan^{-1}x$. It is vital to be familiar with the standard derivatives and integrals provided in the book. Let's work this problem from first principles as it is a useful exercise.

The denominator can't be factorised and this expression can't be simplified using partial fractions. We'll use a trig substitution (which you will almost certainly be given in a question like this one) and another common trig identity.

Let $x = 2\tan u \quad \therefore \; u = \tan^{-1}\left(\frac{x}{2}\right), \qquad \frac{dx}{du} = 2\sec^2 u \quad \therefore \; dx = 2\sec^2 u \, du$

$$\therefore \int \frac{1}{4+x^2} dx = \int \frac{1}{4+4\tan^2 u} 2\sec^2 u \, du = \int \frac{\sec^2 u}{2(1+\tan^2 u)} du \qquad \text{But } 1 + \tan^2 u = \sec^2 u$$

$$\therefore \int \frac{1}{4+x^2} dx = \int \frac{\sec^2 u}{2\sec^2 u} du = \int \frac{du}{2} = \frac{u}{2} + c = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

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15. $\int \frac{8x}{(x^2-4)^5} dx$

Rewriting the denominator in index form may give a clue to the best way to solve this one. The recognition method is works when you can spot that one function . . . $(x^2-4)^{-5}$ . . . is multiplied by a scalar multiple of its derivative . . . $4(2x)$. This suggests that if we differentiated $(x^2-4)^{-4}$, we would probably be close to the answer. Use the chain rule to convince yourself that this is true.

If $\quad y = (x^2-4)^{-4}, \quad \frac{dy}{dx} = -4(2x)(x^2-4)^{-5} = -8x(x^2-4)^{-5}$

$$\therefore \int \frac{8x}{(x^2-4)^5} dx = -\frac{1}{(x^2-4)^4} + c \qquad \text{NB. Subtitute } u = (x^2-4), \text{etc. if in doubt}$$

16. $\int 2\sin 5x \cos 4x \;\, dx$

This is a tricky one which you are unlikely to see at A level unless strong hints are given in the structure of the question about how to start. It uses the double angle formulae:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\sin(A-B) = \sin A \cos B - \cos A \sin B$

Adding together gives: $\quad \sin(A+B) + \sin(A-B) = 2\sin A \cos B$

In reverse: $\quad 2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$$\therefore \int 2\sin 5x \cos 4x \;\, dx = \int \sin(5x+4x) + \sin(5x-4x)\, dx = \int \sin 9x + \sin x \, dx$$

$$\therefore \int 2\sin 5x \cos 4x \;\, dx = -\frac{1}{9}\cos 9x - \cos x + c$$

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17. $\int (x+3)(x+7)^5 dx$

Let's try the substitution method as used in Q.13

Let $u = x+7 \quad \therefore\ x = u-7, \qquad \frac{du}{dx} = 1 \qquad \therefore\ dx = du$

$$\therefore \int (x+3)(x+7)^5 dx = \int (u-7+3)u^5 du = \int (u-4)u^5 du = \int u^6 - 4u^5 du$$

$$\therefore \int (x+3)(x+7)^5 dx = \frac{u^7}{7} - \frac{2u^6}{3} + c = \frac{(x+7)^7}{7} - \frac{2(x+7)^6}{3} + c$$

After some rearranging we have: $\int (x+3)(x+7)^5 dx = \frac{(x+7)^6}{21}(3x+7) + c$

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18. $\int \frac{1+x}{\sqrt{(1-x^2)}} dx$

Very similar to Q.12 and so splitting the expression into two fractions would help.

$$\int \frac{1+x}{\sqrt{(1-x^2)}} dx = \int \left(\frac{1}{\sqrt{(1-x^2)}} + \frac{x}{\sqrt{(1-x^2)}}\right) dx$$

The first fraction is just Q.12 again. The second fraction has a recognition solution as x on top is a scalar derivative of $(1-x^2)$ below. If unsure, make the substitution $u = (1-x^2)$ etc.

$$\therefore \int \frac{1+x}{\sqrt{(1-x^2)}} dx = \sin^{-1} x + (1-x^2)^{\frac{1}{2}} + c$$

19. $\int x \cos x \, dx$

A product of two functions (x and $\cos x$), one of which can be simplified by differentiation (x) and the other which can be integrated, usually suggests that the best method to use is integration by parts.

Recall that for integration by parts: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Let $u = x$ and $\frac{dv}{dx} = \cos x$, $\frac{du}{dx} = 1$ and $v = \sin x$

$$\therefore \int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + c$$

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20. $\int 4x(3x - 5) dx$

Expand the bracket and integrate term by term

$$\int 4x(3x - 5) dx = 12x^2 - 20x \, dx = 4x^3 - 10x^2 + c$$

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21. $\int \frac{10}{25 + x^2} dx$

Use the methods of Q.14

Let $x = 5\tan u$ $\therefore u = \tan^{-1}\left(\frac{x}{5}\right)$, $\frac{dx}{du} = 5\sec^2 u$ $\therefore dx = 5\sec^2 u \, du$

$$\int \frac{10}{25 + x^2} dx = \int \frac{10 \sec^2 u}{5(1 + \tan^2 u)} du = \int \frac{2\sec^2 u}{\sec^2 u} du = \int 2du = 2u + c = 2\tan^{-1}\left(\frac{x}{5}\right) + c$$

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22. $\int 9 \cos 3x \, dx$

Just integrate the $\cos 3x$ and divide by the derivative of $3x$.

$$\int 9 \cos 3x \, dx = \frac{9}{3} \sin 3x + c = 3 \sin 3x + c$$

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23. $\int x^3 \ln x \, dx$

Using the reasoning of Q.19, this is another integration by parts problem. We can only differentiate $\ln x$ so this tells us this must be the u term.

Recall that for integration by parts: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = \ln x$ and $\frac{dv}{dx} = x^3$, $\frac{du}{dx} = \frac{1}{x}$ *and* $v = \frac{x^4}{4}$

$$\therefore \int x^3 \ln x\, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{1}{x}\, dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4} dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + c$$

Which can be simplified to be:

$$\int x^3 \ln x\, dx = \frac{x^4}{16}(4\ln x - 1) + c$$

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24. $\int 2^x dx$

Here we need to rearrange using logarithms:

Let $y = 2^x$, $\ln y = x\ln 2$ $\therefore e^{\ln y} = e^{x\ln 2}$ $\therefore$ $y = e^{x\ln 2}$ This we can integrate

$$\int 2^x dx = \int e^{x\ln 2}\, dx = \frac{e^{x\ln 2}}{\ln 2} + c$$

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25. $\int x\sqrt{(2x+3)}\, dx$

We probably could do this with integration by parts but let's try an obvious substitution instead.

Let $u = 2x + 3$ $\therefore$ $x = \left(\frac{u-3}{2}\right)$, $\frac{du}{dx} = 2$ $\therefore$ $dx = \frac{du}{2}$

$$\int x\sqrt{(2x+3)}\, dx = \int x(2x+3)^{\frac{1}{2}}\, dx = \int \left(\frac{u-3}{2}\right)u^{\frac{1}{2}}\,\frac{du}{2} = \int \frac{1}{4}\left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right) du$$

$$\therefore \int x\sqrt{(2x+3)}\, dx = \frac{1}{4}\left(\frac{2u^{\frac{5}{2}}}{5} - 2u^{\frac{3}{2}}\right) + c = \frac{u^{\frac{5}{2}}}{10} - \frac{u^{\frac{3}{2}}}{2} + c$$

$$\therefore \int x\sqrt{(2x+3)}\, dx = \frac{(2x+3)^{\frac{5}{2}}}{10} - \frac{(2x+3)^{\frac{3}{2}}}{2} + c$$

Which after some factorising and simplifying gives:

$$\int x\sqrt{(2x+3)}\ dx = \frac{(2x+3)^{\frac{3}{2}}}{5}(x-1) + c$$

26. $\int \frac{2x}{x^2+4}\,dx$

I hope you can spot that the derivative of the denominator is in the numerator. This allows us go straight to the answer. Easy !

$$\int \frac{2x}{x^2+4}\,dx \;=\; \ln|x^2+4| + c$$

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27. $\int \frac{6-2x}{(x+1)(x^2+3)}\,dx$

Partial fractions is the only way forward. You should be able to prove that:

$$\int \frac{6-2x}{(x+1)(x^2+3)}\,dx \;=\; \int \left(\frac{2}{(x+1)} - \frac{2x}{(x^2+3)}\right)\,dx$$

$$\therefore \int \frac{6-2x}{(x+1)(x^2+3)}\,dx \;=\; 2\ln|x+1| - \ln|x^2+3| + c \;=\; \ln\left|\frac{(x+1)^2}{(x^2+3)}\right| + c$$

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28. $\int \ln x\,dx$

This integral is often solved in textbooks as a classic test for integration by parts - but you need to use a little lateral thinking beforehand.

$$\int \ln x\,dx = \int 1\,.\ln x\,dx$$

Recall that for integration by parts: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Let $u = \ln x$ and $\frac{dv}{dx} = 1,\quad \frac{du}{dx} = \frac{1}{x}\;$ *and* $\;v = x$

$$\therefore \int \ln x\,dx = x\ln x - \int x\frac{1}{x}\,dx \;=\; x\ln x - \int dx \;=\; x\ln x - x + c \;= x(\ln x + 1) + c$$

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29. $\int 30\cos 4x \cos x\,dx$

Follow the procedure in Q.16, except using cosines.

If $\;2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\cos 4x\cos x = \cos 5x + \cos 3x$

$$\therefore \int 30\cos 4x\cos x\,dx = \int 15\,(\cos 5x + \cos 3x)\,dx \;=\; 3\sin 5x + 5\sin 3x + c$$

30. $\int 4\sec^2 x\, dx$

This is a standard integral from the formula book. Let's try to solve it anyway. Whenever you see $\sec^2 x$, look out for a $\tan x$.

$$\text{Let } u = \tan x, \quad \frac{du}{dx} = \sec^2 x \quad \therefore\ dx = \frac{du}{\sec^2 x}$$

$$\int 4\sec^2 x\, dx = \int 4\sec^2 x \frac{du}{\sec^2 x} = \int 4\, du = 4u + c = 4\tan x + c$$

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31. $\int \frac{1}{\sqrt{(x+5)}}\, dx$

Should be a simple problem if you've made it this far.

$$\int \frac{1}{\sqrt{(x+5)}}\, dx = \int (x+5)^{-\frac{1}{2}}\, dx = 2(x+5)^{\frac{1}{2}} + c$$

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32. $\int \sin^7 x\, dx$

Not unlike Q.9. It's an ODD power so we need to use a trig identity to turn the integral into one which we can integrate.

$$\int \sin^7 x\, dx = \int \sin x \sin^6 x\, dx = \int \sin x\, (1-\cos^2 x)^3\, dx$$

Expanding the last bracket: $(1-\cos^2 x)^3 = (1 - 3\cos^2 x + 3\cos^4 x - \cos^6 x)$

$$\therefore \int \sin^7 x\, dx = \int (\sin x - 3\sin x\cos^2 x + 3\sin x\cos^4 x - \sin x\cos^6 x)\, dx$$

$$\therefore \int \sin^7 x\, dx = -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + c$$

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33. $\int \cos^4 2x\, dx$

Unlike Qs. 9 and 32, the trig function power here is EVEN. The rule followed in this event is to use double angle trig identities to remove the 4th power completely, leaving terms which can be integrated directly. Recall:

$$\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2} \qquad \therefore \cos^2 2x = \frac{1}{2} + \frac{\cos 4x}{2}$$

$$\therefore \int \cos^4 2x\,dx = \int (\cos^2 2x)^2 dx = \int \left(\frac{1}{2}+\frac{\cos 4x}{2}\right)^2 dx = \int \left(\frac{1}{4}+\frac{\cos 4x}{2}+\frac{\cos^2 4x}{4}\right) dx$$

But $\cos^2 4x = \frac{1}{2}+\frac{\cos 8x}{2}$

$$\therefore \int \cos^4 2x\,dx = \int \left(\frac{1}{4}+\frac{\cos 4x}{2}+\frac{1}{4}\left(\frac{1}{2}+\frac{\cos 8x}{2}\right)\right) dx = \int \left(\frac{3}{8}+\frac{\cos 4x}{2}+\frac{\cos 8x}{8}\right) dx$$

$$\therefore \int \cos^4 2x\,dx = \frac{3x}{8}+\frac{\sin 4x}{8}+\frac{\sin 8x}{64}+c$$

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34. $\int \frac{\sin x}{\cos^2 x} dx$

Once again, a simple substitution should do the trick.

Let $u = \cos x$ $\quad \frac{du}{dx} = -\sin x \quad \therefore\ dx = -\frac{du}{\sin x}$

$$\int \frac{\sin x}{\cos^2 x} dx = -\int \frac{\sin x}{u^2}\frac{du}{\sin x} = -\int \frac{du}{u^2} = \frac{1}{u}+c = \frac{1}{\cos x}+c = \sec x + c$$

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35. $\int \frac{x}{x+10} dx$

Sometimes all you need is a little algebra to rearrange to give something easy to integrate. Adding and subtracting a number in the numerator can quickly simplify fractions.

$$\int \frac{x}{x+10} dx = \int \frac{x+10-10}{x+10} dx = \int \left(\frac{x+10}{x+10}-\frac{10}{x+10}\right) dx = \int \left(1-\frac{10}{x+10}\right) dx$$

$$\therefore \int \frac{x}{x+10} dx = x - 10\ln|x+10| + c$$

There are 56 questions in Ex 20C of Understanding Pure Mathematics. I'll only show workings of a few more selected questions which illustrate some interesting points. They are generally more difficult than Qs. 1-35.

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44. $\int \sin^{-1}\left(\frac{x}{3}\right) dx$

This is standard integral but let's try to solve it from first principles. Remember Q.12 showed that we could obtain $\sin^{-1} x$ by integrating a function. It follows that we could differentiate $\sin^{-1} x$ and get the same original function. Look back at Q.28 and see how we used the fact that we could differentiate $\ln x$ in order to integrate $\ln x$ using integration by parts. We'll use the same strategy here.

$$\int \sin^{-1}\left(\frac{x}{3}\right) dx \;=\; \int 1.\sin^{-1}\left(\frac{x}{3}\right) dx$$

Recall that for integration by parts: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = \sin^{-1}\left(\frac{x}{3}\right)$ and $\frac{dv}{dx} = 1$, $\quad \frac{du}{dx} = \frac{1}{\sqrt{(9-x^2)}}$ and $v = x$

$$\therefore \int \sin^{-1}\left(\frac{x}{3}\right) dx \;=\; x\sin^{-1}\left(\frac{x}{3}\right) - \int \frac{x}{\sqrt{(9-x^2)}}\,dx$$

Taking this new integral, $\int \frac{x}{\sqrt{(9-x^2)}}\,dx$, let's use a substitution to solve it

Let $p = (9-x^2)$, $\quad \frac{dp}{dx} = -2x \quad \therefore \quad dx = -\frac{dp}{2x}$

$$\therefore \int \frac{x}{\sqrt{(9-x^2)}}\,dx \;=\; -\int \frac{x}{p^{\frac{1}{2}}}\frac{dp}{2x} \;=\; -\int \frac{p^{-\frac{1}{2}}}{2}dp \;=\; -p^{\frac{1}{2}} + c \;=\; -\sqrt{(9-x^2)} + c$$

Therefore the whole integral now becomes:

$$\therefore \int \sin^{-1}\left(\frac{x}{3}\right) dx \;=\; x\sin^{-1}\left(\frac{x}{3}\right) + \sqrt{(9-x^2)} + c$$

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51. $\int \tan^3 x\,dx$

Odd powered trig functions usually solve by factorising and using identities in order to set up functions which can be integrated, as we saw in Q9 and Q32.

$$\int \tan^3 x\,dx \;=\; \int \tan x \tan^2 x\,dx \;=\; \int \tan x\,(\sec^2 x - 1)dx \;=\; \int \tan x \sec^2 x - \tan x\,dx$$

You should recognise that $\sec^2 x$ is the derivative of $\tan x$ which tells us the form of the solution of the first term (i.e. $\tan^2 x$). We'll use a substitution to demonstrate this. The second term is a standard integral.

$\int \tan x \sec^2 x\,dx \quad$ Let $u = \tan x$, $\quad \frac{du}{dx} = \sec^2 x \quad \therefore \quad dx = \frac{du}{\sec^2 x}$

$$\therefore \int \tan x \sec^2 x\,dx = \int u \sec^2 x\frac{du}{\sec^2 x} \;=\; \int u\,du = \frac{u^2}{2} + c \;=\; \frac{\tan^2 x}{2} + c$$

Therefore the whole integral now becomes:

$$\int \tan^3 x\, dx \ = \frac{\tan^2 x}{2} - \ln|\sec x| + c$$

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55. $\int \frac{2}{x^2 + 2x + 5} dx$

The denominator doesn't factorise and the numerator doesn't contain the derivative of the denominator. Remember the completing the square procedure ? What does that give us ?

$$\int \frac{2}{x^2 + 2x + 5} dx \quad = \quad \int \frac{2}{(x+1)^2 + 4} dx$$

This is a bit like the standard integral we saw in Q.14, namely:

$$\int \frac{1}{x^2+a^2} dx \ = \ \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

In this case: $x = x + 1$ and $a = 2$

$$\therefore \int \frac{2}{x^2 + 2x + 5} dx \ = \ 2\left[\frac{1}{2} \tan^{-1}\left(\frac{x+1}{2}\right)\right] + c \ = \ \tan^{-1}\left(\frac{x+1}{2}\right) + c$$

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00. $\int e^x \ \sin x\, dx$

This integral is an extra one I've added because it illustrates the method for dealing with the product of two functions which don't simplify when tackled using integration by parts. Let's use that method anyway and see where it takes us.

Recall that for integration by parts: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Let $u = e^x$ and $\frac{dv}{dx} = \sin x,$ $\quad \frac{du}{dx} = e^x \ \ and \ \ v = -\cos x$

$$I = \int e^x \ \sin x\, dx \ = \ -e^x \ \cos x - \int -\cos x \ e^x\, dx \ = \ -e^x \ \cos x + \int e^x \cos x \ \ dx$$

We now have another integral to solve by parts (in red):

Let $u = e^x$ and $\frac{dv}{dx} = \cos x,$ $\quad \frac{du}{dx} = e^x \ \ and \ \ v = \sin x$

$$\int e^x \cos x \ \ dx \ = \ e^x \ \sin x - \int e^x \ \sin x \ dx \ = \ e^x \ \sin x - I$$

Note that we have obtained the original integral, I , as an output of our second integration by parts procedure. Putting it all together now:

$$I = \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - I$$

Adding I to both sides gives:

$$2I = e^x (sin \ x - cos \ x)$$

$$\therefore \ I = \frac{e^x}{2} (sin \ x - cos \ x) + c$$
