The Special Theory of Relativity: Why does E = mc² ?

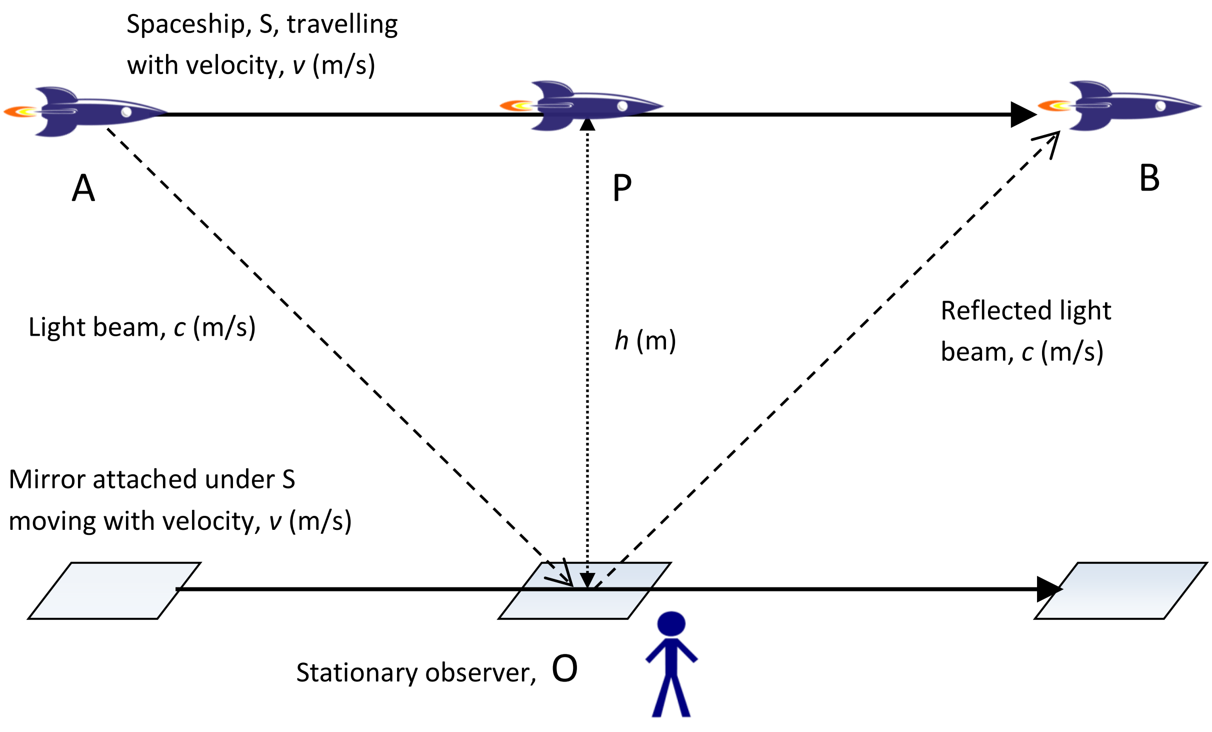
Adapted from "*A Most Incomprehensible Thing*" by Peter Collier, 2012

Two postulates:

1) Principal of Relativity - the laws of physics are the same in any inertial frame of reference regardless of velocity
2) The speed of light is constant at "c" for all observers

Time Dilation

A spaceship S emits a beam of light which is reflected back to a receiver on the ship via a mirror fixed h metres underneath the ship.



From the spaceship S, the time taken, τ, for the light beam to be reflected is the distance PO plus OP divided by the velocity of light, c.

$$\tau = \frac{2h}{c}$$

From the observer's viewpoint, the time taken, t, is also distance divided by c but the observed distance is different: AO plus OB (i.e. longer than PO plus OP)

$$AB = vt \quad \rightarrow \quad AP = \frac{AB}{2} = \frac{vt}{2}$$

Using Pythagoras, we have

$$PO = h \quad \rightarrow \quad AO = \sqrt{h^2 + \left(\frac{vt}{2}\right)^2}$$

Therefore, the distance travelled by the light beam as measured by the observer is

$$AO + OB = 2\sqrt{h^2 + \left(\frac{vt}{2}\right)^2}$$

The time taken for the route AO plus OB is

$$t = \frac{2\sqrt{h^2 + \left(\frac{vt}{2}\right)^2}}{c}$$

Now make t the subject of this last equation

$$ct = 2\sqrt{h^2 + \left(\frac{vt}{2}\right)^2}$$

$$\frac{c^2t^2}{4} = h^2 + \frac{v^2t^2}{4}$$

$$\frac{t^2(c^2 - v^2)}{4} = h^2$$

$$t = \frac{2h}{\sqrt{c^2 - v^2}} = \frac{2h}{c\sqrt{1 - \frac{v^2}{c^2}}}$$

But we showed that $\tau = \dfrac{2h}{c}$

Therefore $\quad t = \dfrac{\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\tau \quad$ Where $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Since $v > 0$ it follows that $\gamma > 1$ and therefore $t > \tau$!!!!!

In other words, the time taken for the light beam to bounce off the mirror takes longer from the point of view of the observer than it does from the point of view of the spaceship. **Time appears to be running slower on the spaceship than it does on the ground, as measured by the observer.** This phenomenon is known as time dilation. In relativity, we need to modify Newtonian mechanics to take account of time dilation.

<u>Relativistic Momentum</u>

Newtonian momentum of a moving body, $p_N = mv = m \frac{dx}{d\tau}$

But $dt = \gamma \, d\tau$ So relativistic momentum,

$$p_R = m \, \gamma \frac{dx}{dt} = m \, \gamma \, v = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where $m = \text{rest mass of the body}$

<u>Kinetic Energy: Newtonian then Relativistic</u>

Work done accelerating a particle from over the distance from x_0 to x_1 is given by

$$W = \int_{x_0}^{x_1} F \, dx$$

Where $F = ma$ and $a = \frac{dv}{dt}$

Therefore

$$W = \int_{x_0}^{x_1} ma \, dx = \int_{x_0}^{x_1} m \frac{dv}{dt} \, dx$$

Using the chain rule, we can now express this work as a function of initial and final velocities

$$\frac{dv}{dt} \, dx = \frac{dv}{dx} \frac{dx}{dt} \, dx$$

$$W = \int_{x_0}^{x_1} m \frac{dv}{dx} \frac{dx}{dt} \, dx = \int_{v_0}^{v_1} mv \, dv$$

Integrating this gives

$$W = \left[\frac{mv^2}{2}\right]_{v_0}^{v_1} = \frac{m{v_1}^2}{2} - \frac{m{v_0}^2}{2}$$

If $v_0 = 0$, Newtonian kinetic energy $W = KE_N = \frac{mv^2}{2}$

Recall that $p_N = mv$, therefore

$$W = \int_{v_0}^{v_1} mv\, dv = \int_{p_0}^{p_1} v\, dp$$

where p_0 and p_1 are the body's initial and final momentum

<u>But</u> also recall that

$$p_R = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore, relativistic kinetic energy is given by

$$KE_R = \int_{v_0}^{v_1} v\, d\left(\frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

Solve using integration by parts

$\int \alpha \, \frac{d\beta}{dv} dv = \alpha\beta - \int \beta \, \frac{d\alpha}{dv} dv$ where $\alpha = v, \quad \beta = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$KE_R = \left[\frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}}\right]_{v_0}^{v_1} - m\int_{v_0}^{v_1} \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\, dv$$

Integrating the second part of this integral using a substitution, s

$I = m \int_{v_0}^{v_1} \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\, dv$ Let $s = 1 - \frac{v^2}{c^2} \rightarrow \frac{ds}{dv} = \frac{-2v}{c^2} \rightarrow dv = \frac{-c^2}{2v} ds$

$$I = -m\int_{v_0}^{v_1} \frac{vc^2}{2v\sqrt{s}}\, ds = -\frac{mc^2}{2}\int_{v_0}^{v_1} s^{-\frac{1}{2}}\, ds$$

$$I = -\frac{mc^2}{2}\left[2s^{\frac{1}{2}}\right]_{v_o}^{v_1} = -mc^2\left[s^{\frac{1}{2}}\right]_{v_0}^{v_1} = -mc^2\left[\sqrt{1-\frac{v^2}{c^2}}\right]_{v_0}^{v_1}$$

So now put this result back into the expression for relativistic kinetic energy

$$KE_R = \left[\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}\right]_{v_0}^{v_1} + mc^2\left[\sqrt{1-\frac{v^2}{c^2}}\right]_{v_0}^{v_1}$$

$$KE_R = \left[\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{mc^2(1-\frac{v^2}{c^2})}{\sqrt{1-\frac{v^2}{c^2}}}\right]_{v_0}^{v_1} = \left[\frac{mv^2 + mc^2 - mc^2\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right]_{v_0}^{v_1}$$

$$KE_R = \left[\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right]_{v_0}^{v_1} = \frac{mc^2}{\sqrt{1-\frac{{v_1}^2}{c^2}}} - \frac{mc^2}{\sqrt{1-\frac{{v_0}^2}{c^2}}}$$

But when accelerating from rest, $v_0 = 0,\ v_1 = v$

$$KE_R = mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$$

This important result shows that the relativistic kinetic energy of a moving body can be expressed as two terms, the first of which depends on both the body's mass and velocity and the second depends only on the body's mass. In other words, the relativistic kinetic energy is the total energy of that body minus the energy of that body when it is stationary.

But is this new KE_R really the kinetic energy of a body? Let's check to see if it reduces to KE_N at slow velocities.

We can use Taylor's expansion for the term $1/\sqrt{1-\frac{v^2}{c^2}}$

$$\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2+\cdots$$

So, we can express the relativistic kinetic energy as

$$KE_R = mc^2\left[\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2+\cdots\right)-1\right]$$

When $v \ll c$ this approximates to the Newtonian kinetic energy

$$KE_R \approx KE_N = \frac{mv^2}{2}$$

Therefore, the total energy of the moving body is its kinetic energy plus its energy at rest

$$E = KE_R + mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma mc^2$$

But when $v = 0,\ \gamma = 1$

Therefore, the intrinsic energy of a body associated only with its rest mass is

$$\boldsymbol{E = mc^2}$$